Quadri-tilings of the plane

de Tilière, B

Abstract: We introduce quadri-tilings and show that they are in bijection with dimer models on a family of graphs R * arising from rhombus tilings. Using two height functions, we interpret a sub-family of all quadri-tilings, called triangular quadri-tilings, as an interface model in dimension 2+2. Assigning "critical" weights to edges of R *, we prove an explicit expression, only depending on the local geometry of the graph R *, for the minimal free energy per fundamental domain Gibbs measure; this solves a conjecture of Kenyon (Invent Math 150:409–439, 2002). We also show that when edges of R * are asymptotically far apart, the probability of their occurrence only depends on this set of edges. Finally, we give an expression for a Gibbs measure on the set of all triangular quadri-tilings whose marginals are the above Gibbs measures, and conjecture it to be that of minimal free energy per fundamental domain.

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Quadri-tilings of the plane

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Abstract

Quadri-tilings of the plane are tilings by quadrilaterals made of adjacent right triangles. Quadri-tilings are in bijection with dimer configurations on graphs arising from rhombus tilings of the plane. Assigning "critical" weights to the edges of such a graph, we construct a natural explicit Gibbs measure, and prove that it is asymptotically independent of the structure of the graph. We give an explicit expression for a measure on the set of dimer configurations of all graphs arising from 60°-rhombus tilings of the plane, whose marginals are the above Gibbs measures. We construct two "height functions" on 60°-rhombus quadri-tilings, and thereby interpret them as surfaces in a 4-dimensional space.

1 Introduction

Consider the set of right triangles whose hypotenuses have length two. Colour the vertex at the right angle black, and the other two vertices white. A **quadri-tile** is a quadrilateral obtained from two such triangles in two different ways: either glue them along the hypotenuse, or supposing they have a leg of the same length, glue them along this edge matching the black (white) vertex to the black (white) one, see figure 1. Both types of quadri-tiles have four vertices.

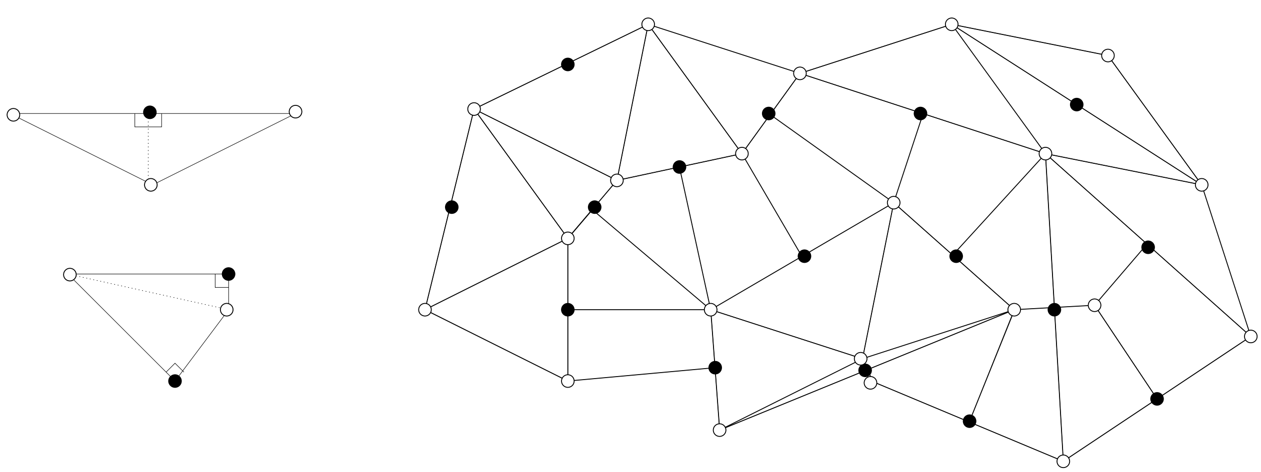

Figure 1: Two types of quadri-tiles (left), and a quadri-tiling (right).

A **quadri-tiling** of the plane is an edge-to-edge tiling of the plane by quadri-tiles that respects the colouring of the vertices, that is black (resp. white) vertices are matched to black (resp. white)

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ones. An example of quadri-tiling is given in figure 1. We are concerned in this paper with $\mathcal{Q}$, the set of all quadri-tilings of the plane that use finitely many different quadri-tiles up to isometry.

Section 2 describes features of quadri-tilings. Let $\mathcal{R}$ be the set of all side-length two, edge-to-edge, rhombus tilings of the plane that have only finitely many different rhombus angles. If $R \in \mathcal{R}$ is a rhombus tiling of the plane, denote by R_d the tiling of the plane obtained from R by adding the diagonals of the rhombi (faces of the graph R_d are right triangles). Let $\mathcal{Q}(R_d)$ be the set of all quadri-tilings of the plane refined by R_d, that is quadri-tiles are made of adjacent right triangles of R_d, in such a way that if each tile of a quadri-tiling in $\mathcal{Q}(R_d)$ is divided into its two triangles, we obtain R_d. In section 2.1, we prove that to every quadri-tiling $T \in \mathcal{Q}$, there corresponds a unique rhombus tiling $R(T) \in \mathcal{R}$ with the property that $T \in \mathcal{Q}(R_d(T))$. $R(T)$ is called the **underlying rhombus tiling** of T.
In section 2.2, for a fixed rhombus-with-diagonals tiling R_d, we define a $\mathbb{Z}$-valued function h_1, called the **first height function**, on the vertices of every quadri-tiling $T \in \mathcal{Q}(R_d)$ (which are the same as the vertices of R_d), and prove that quadri-tilings in $\mathcal{Q}(R_d)$ are in bijection with first height functions defined on the vertices of R_d.
We define **lozenges** to be $60°$-rhombi, and we denote by $\mathcal{L} \subset \mathcal{R}$ the set of lozenge tilings of the plane up to isometry. We denote by $\mathcal{Q}'$ the set of all quadri-tilings with underlying lozenge tilings, that is $\mathcal{Q}' = \cup_{L\in\mathcal{L}}\mathcal{Q}(L_d)$. To every quadri-tiling $T \in \mathcal{Q}'$, we assign the height function h_1, and Thurston's lozenge height function [15] denoted by h_2, and we deduce that quadri-tilings of $\mathcal{Q}'$ are characterized by h_1 and h_2. Based on h_1 and h_2, we give a geometric interpretation of quadri-tilings of $\mathcal{Q}'$ as surfaces in a 4-dimensional space that have been projected on the plane. In section 2.3, we give elementary operations that allow to transform any quadri-tiling of a simply connected region into any other.

Quadri-tilings correspond bijectively to dimer configurations in the following way. If G is a graph, denote by G^* the **dual graph** associated to G: faces of G correspond to vertices of G^*, and two vertices of G^* are connected by an edge if the corresponding faces of G are adjacent. We denote by f^* the dual vertex of a face f, and by e^* the dual edge of an edge e. A **dimer configuration** of a graph G (also called **perfect matching**) is a subset of edges M of the graph which has the property that every vertex of G is incident to exactly one edge of M. Let us denote by $\mathcal{M}(G)$ the set of dimer configurations of the graph G. Quadri-tilings of R_d are in bijection with dimer configurations of ${R_d}^*$. Indeed, if $T \in \mathcal{Q}(R_d)$ is a quadri-tiling of the plane, associate to every quadri-tile made of adjacent right triangles f and g, the edge f^*g^* of ${R_d}^*$. The set of edges of ${R_d}^*$ corresponding to the set of quadri-tiles of T form a perfect matching. We let $\mathcal{M} = \cup_{R\in\mathcal{R}}\mathcal{M}({R_d}^*)$, and in the case of lozenges we let $\mathcal{M}' = \cup_{L\in\mathcal{L}}\mathcal{M}({L_d}^*)$. In statistical physics a dimer configuration represents a system of diatomic molecules (dimer) adsorbed on the surface of a crystal [6].

Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, and assume a weight function ν is associated to the edges of ${R_d}^*$. Consider a finite simply connected subgraph R_d^{1*} of ${R_d}^*$. Then there is a natural measure μ^{R^1}, called the **Boltzmann measure**, defined on $\mathcal{M}(R_d^{1*})$ by

$$\mu^{R^1}(M) = \frac{\prod_{e\in M}\nu(e)}{Z(R_d^{1*},\nu)},$$

where $Z(R_d^{1*},\nu) = \sum_{M\in\mathcal{M}(R_d^{1*})}\prod_{e\in M}\nu(e)$ is the **dimer partition function**.

A **Gibbs measure** on $\mathcal{M}({R_d}^*)$ is defined to be a probability measure with the following property [9]. If the matching in an annular region is fixed, the matchings inside and outside of the annulus are independent of each other, and the probability of any interior matching M is proportional to $\prod_{e\in M}\nu(e)$.

In section 3, for a specific weight function, we construct a natural explicit Gibbs measure μ^R on $\mathcal{M}({R_d}^*)$. More precisely, the graph R_d and its dual ${R_d}^*$ satisfy a geometric condition called isoradiality, so that we assign the "critical" weight function [7] to the edges of ${R_d}^*$ (see also section 3.1). For this weight function, the **Dirac operator** is represented by the **Kasteleyn matrix** K which is indexed by the vertices of ${R_d}^*$. The existence and uniqueness of the inverse Dirac operator K^{-1} is proved in [7] (see also section 3.2). The graph ${R_d}^*$ is bipartite (see lemma 2.2), that is vertices of ${R_d}^*$ can be divided into two subsets $B \cup W$, and vertices in B (the black vertices) are only adjacent to vertices in W (the white vertices). If $\{w_1 b_1, \ldots, w_k b_k\}$ is a subset of edges of ${R_d}^*$, denote by E the set of dimer configurations in $\mathcal{M}({R_d}^*)$ that contain these edges.

Theorem 1.1 *There exists a unique probability measure μ^R on $\mathcal{M}({R_d}^*)$ that satisfies*

$$\mu^R(E) = \left(\prod_{i=1}^{k} K(w_i, b_i)\right) \det_{1 \leq i,\, j \leq k} \left(K^{-1}(b_i, w_j)\right), \tag{1}$$

moreover μ^R is a Gibbs measure on $\mathcal{M}({R_d}^*)$.

(See also theorem 3.2). The Gibbs measure μ^R is a natural extension of the Boltzmann measure to the infinite graph (see remark 3.3). Theorem 1.1 is proved by showing convergence to (1) of the Boltzmann measure on some appropriate torus. The construction of the torus relies on the fact that any finite simply connected sub-graph of a rhombus tiling of the plane can be embedded in a periodic rhombus tiling of the plane (proposition 3.4). The convergence argument (proposition 3.8) uses a result of [9] on the convergence of the inverse Kasteleyn matrix in the case of periodic graphs, and a locality property of the inverse Dirac operator in the case of critical weights.

In section 4, we restrict ourselves to quadri-tilings with underlying lozenge tilings, and we construct a measure μ on $\mathcal{M}'$. Denote by H the honeycomb lattice, then lozenge tilings correspond to dimer configurations of H. Let ν be a measure on $\mathcal{M}(H)$ (see [8] for an example of such a measure). If $\{e_1, \ldots, e_k\}$ is a subset of edges of $\cup_{L \in \mathcal{L}} {L_d}^*$, denote by $\mathcal{E}$ the set of dimer configurations in $\mathcal{M}'$ that contain these edges. Refer to section 4 for definitions below. Assume that the lozenges associated to the edges $\{e_1, \ldots, e_k\}$ form a connected path, and denote by $\{f_1, \ldots, f_l\}$ the dual edges of H corresponding to these lozenges.

Theorem 1.2 *There exists a unique probability measure μ on $\mathcal{M}'$ that satisfies*

$$\mu(\mathcal{E}) = \nu(F)\mu^L(E), \tag{2}$$

where F is the cylinder set of H corresponding to the edges $\{f_1, \ldots, f_l\}$, $L \in \mathcal{L}$ is a lozenge tiling of the plane that contains the lozenges associated to the edges $e_1, \ldots, e_k$, and E is the cylinder set of ${L_d}^*$ corresponding to the edges $e_1, \ldots, e_k$.

Let $R \in \mathcal{R}$ be a rhombus tiling of the plane. In section 5, we determine the asymptotics of the inverse Dirac operator indexed by the vertices of ${R_d}^*$. For a general isoradial graph, an asymptotic formula for $K^{-1}(b, w)$, as $|b - w| \to \infty$ is given in [7] (see also equation(14)); $K^{-1}(b, w)$ depends on the angles of an edge-path from w to b. In the case where the isoradial graph is ${R_d}^*$, we prove that asymptotically $K^{-1}(b, w)$ only depends on the first and last angle of the edge-path.

Theorem 1.3

$$K^{-1}(b, w) = \frac{1}{2\pi}\left(\frac{1}{b-w} + \frac{e^{-i(\theta_1+\theta2)}}{\bar{b}-\bar{w}}\right) + \frac{1}{2\pi}\left(\frac{e^{2i\theta_1} + e^{2i\theta_2}}{(b-w)^3} + \frac{e^{-i(3\theta_1+\theta_2)} + e^{-i(\theta_1+3\theta_2)}}{(\bar{b}-\bar{w})^3}\right) + O\left(\frac{1}{|b-w|^3}\right).$$

(See also theorem 5.1). As a consequence, we deduce that when the edges of E are asymptotically far apart, the Gibbs measure μ^R is independent of the underlying rhombus tiling R (see corollary 5.2). We conclude by giving a consequence of corollary 5.2 for the measure μ.

Acknowledgments: we would like to thank Richard Kenyon for proposing the quadri-tile dimer model and asking the questions related to it. We would also like to thank him for the many enlightening discussions.

2 Features of quadri-tilings

2.1 Underlying rhombus tilings

The following lemma gives a bijection between quadri-tilings of the plane and quadri-tilings refined by rhombus-with-diagonals tilings of the plane.

Lemma 2.1 $\mathcal{Q} = \bigcup_{R\in\mathcal{R}} \mathcal{Q}(R_d)$.

Proof: Consider a quadri-tiling of the plane $T \in \mathcal{Q}$. Denote by R_d the tiling of the plane obtained from T by drawing, for each quadri-tile, the edge separating the two right triangles. Let b be a black vertex of R_d, denote by $w_1, \ldots, w_k$ the neighbours of b in counterclockwise order. In each right triangle, the black vertex is adjacent to two white vertices, and since the gluing respects the colouring of the vertices, $w_1, \ldots, w_k$ are white vertices. Moreover, b is at the right angle, so $k = 4$ and the edges w_1w_2, w_2w_3, w_3w_4, w_4w_1 are hypotenuses of right triangles. Therefore $w_1, \ldots, w_4$ form a side-length two rhombus, and b stands at the crossing of its diagonals. This is true for any black vertex b of R_d, so R_d is a rhombus-with-diagonals tiling of the plane, and using the definition of $\mathcal{Q}(R_d)$ we conclude that $T \in \mathcal{Q}(R_d)$. □

Recall that for a quadri-tiling of the plane $T \in \mathcal{Q}$, the rhombus tiling $R(T)$ such that $T \in \mathcal{Q}(R_d(T))$ is called the **underlying rhombus tiling** of T.

2.2 Height functions

2.2.1 The first height function

Consider a rhombus tiling of the plane $R \in \mathcal{R}$, and a quadri-tiling of the plane $T \in \mathcal{Q}(R_d)$. In this section, we define a $\mathbb{Z}$-valued function h_1 on the vertices of T (which are the same as the vertices of R_d), called the **first height function**. Lemma 2.3 gives a bijection between quadri-tilings in $\mathcal{Q}(R_d)$ and first height functions defined on the vertices of R_d. To define h_1, we need a bipartite colouring of the faces of R_d which is given by the following.

Lemma 2.2 *Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, then R_d has a black and white bipartite colouring of its faces, which is also a bipartite colouring of the vertices of ${R_d}^*$.*

Proof: The cycles corresponding to the faces of the graph R have length four, thus R has a black and white bipartite colouring of its vertices. Note that faces of the graph R_d are right triangles whose hypotenuse is a rhombus edge. Consider such a face and orient its boundary edges counterclockwise. If the white vertex of the hypotenuse-edge comes before the black one, assign colour black to the face, else assign colour white. This defines a bipartite colouring of the faces of R_d (see figure 2).

□

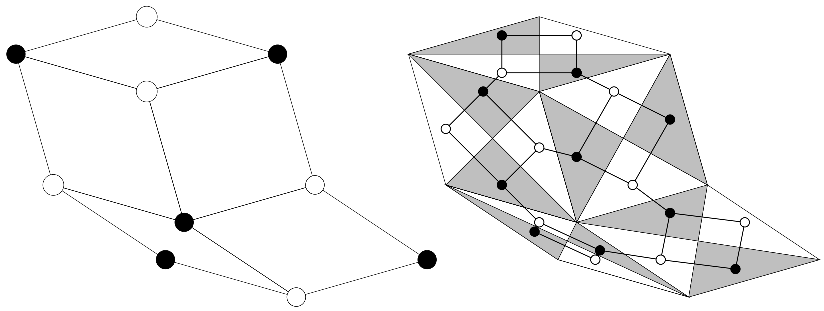

Figure 2: Bipartite colouring of the vertices of R (left), and corresponding bipartite colouring of the faces of R_d and of the vertices of ${R_d}^*$ (right).

From now on we write cclw for counterclockwise, and cw for clockwise.
Orient the edges around the black faces of the graph R_d cclw, the edges around the white faces are then oriented cw, and define h_1 on the vertices of a quadri-tiling $T \in \mathcal{Q}(R_d)$ as follows. Choose a vertex v_1 on a boundary edge of a rhombus of R, and set $h_1(v_1) = 0$. For every other vertex v of T, take an edge-path γ_1 from v_1 to v which follows the boundaries of the quadri-tiles of T. The first height function h_1 changes by ± 1 along each edge of γ_1: if an edge is oriented in the direction of the path, then h_1 increases by 1, if it is oriented in the opposite direction, then h_1 decreases by 1. $h_1(v)$ is independent of the path γ_1 because the plane is simply connected, and the height change around any quadri-tile is zero. An example of computation of h_1 is given in figure 3.

Lemma 2.3 *Let $R \in \mathcal{R}$ be a rhombus tiling of the plane. Fix a vertex v_1 on a boundary edge of a rhombus of R. Let $\tilde{h_1}$ be a $\mathbb{Z}$-valued function on the vertices of R_d satisfying the following two conditions:*

- $\tilde{h_1}(v_1) = 0$,
- $\tilde{h_1}(v) = \tilde{h_1}(u) + 1$, *or* $\tilde{h_1}(v) = \tilde{h_1}(u) - 2$, *for any edge uv oriented from u to v.*

Then, there is a bijection between functions $\tilde{h_1}$ satisfying these two conditions, and quadri-tilings in $\mathcal{Q}(R_d)$.

Proof: The idea of the proof closely follows [3].
If $T \in \mathcal{Q}(R_d)$ is a quadri-tiling of the plane, the first height function h_1 satisfies the two conditions of the lemma: if an edge uv, oriented from u to v, belongs to the boundary of a quadri-tile, it satisfies $h_1(v) = h_1(u) + 1$, else if it lies across a quadri-tile, it satisfies $h_1(v) = h_1(u) - 2$.
Let $\tilde{h_1}$ be a $\mathbb{Z}$-valued function as in the lemma. Then, anytime there is an edge uv satisfying $|\tilde{h_1}(v) - \tilde{h_1}(u)| = 2$, put a quadri-tile made of the two right triangles adjacent to this edge. This defines a quadri-tiling of the plane in $\mathcal{Q}(R_d)$. □

2.2.2 The second height function

Let $T \in \mathcal{Q}'$ be a quadri-tiling of the plane, and let $L(T)$ be its underlying lozenge tiling. We assign the first height function to the vertices of T. In this section, following Thurston [15], we define a second $\mathbb{Z}$-valued function h_2 on the vertices of $L_d(T)$ (which are the same as the vertices of T), called the **second height function**. We also give a geometric interpretation of h_1 and h_2.

We obtain the equilateral triangular lattice by dividing each lozenge of $L(T)$ in two along the bisector of the $60°$ angle. The triangular lattice has a black and white bipartite colouring of its faces. Orient the edges around the black faces cclw, the edges around the white faces are then oriented cw.
Thurston [15] defines h_2 as follows: choose a vertex v_2 of $L(T)$, and set $h_2(v_2) = 0$. For every other vertex v of $L(T)$, take an edge-path γ_2 from v_2 to v which follows the boundaries of the lozenges of $L(T)$. The second height function h_2 changes by ± 1 along each edge of γ_2: if an edge is oriented in the direction of the path, then h_2 increases by 1, if it is oriented in the opposite direction, then h_2 decreases by 1. $h_2(v)$ is independent of the path γ_2 because the plane is simply connected, and the height change around any lozenge is zero. For convenience, we choose v_2 to be the same vertex as v_1, and denote this common vertex by v_0, so that $h_1(v_0) = h_2(v_0) = 0$. An analog to lemma 2.3 gives a bijection between second height functions and lozenge tilings of the plane. Thus, recalling that $\mathcal{Q}' = \cup_{L \in \mathcal{L}} \mathcal{Q}(L_d)$, we deduce that quadri-tilings in $\mathcal{Q}'$ are characterized by h_1 and h_2.
Let us define a natural value for the second height function at the vertex in the center of the lozenges of $L(T)$. When going cclw around the vertices of a lozenge l of $L(T)$, starting from the smallest value of h_2 say h, vertices take on successive values $h, h+1, h+2, h+1$, so that we give value $h+1$ at the vertex in the center of the lozenge l. An example of computation of h_2 is given in figure 3.

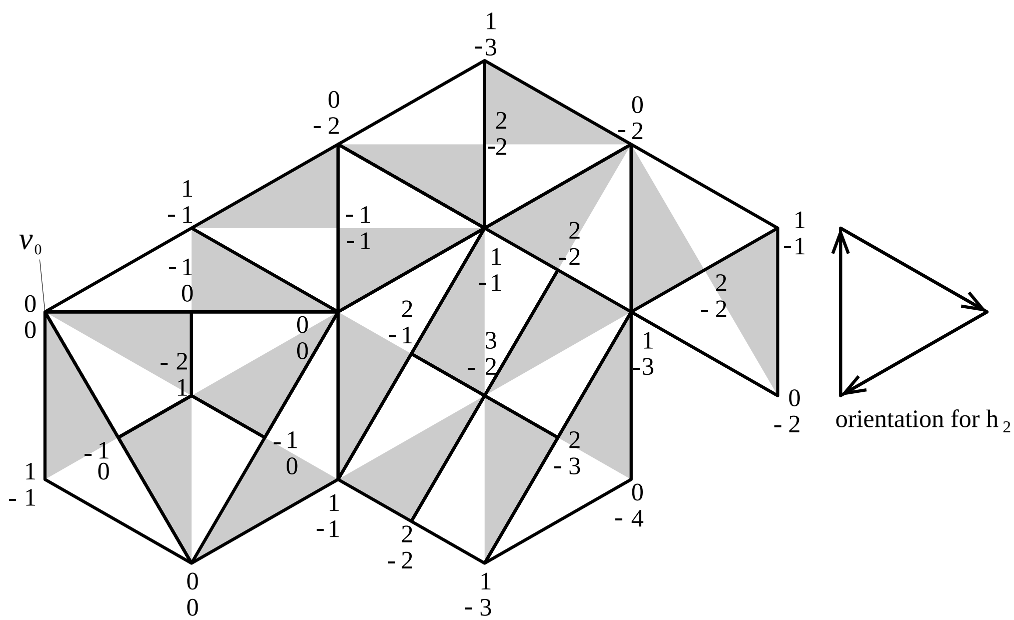

Figure 3: Quadri-tiling of an underlying lozenge tiling with height functions h_1 (above) and h_2 (below).

In Thurston's geometric interpretation [15], a rhombus tiling is seen as a surface S in $\mathbb{Z}^3$ (where the diagonals of the cubes are orthogonal to the plane) that has been projected orthogonally on the plane. S is determined by the height function h_2. In a similar way, a quadri-tiling of the plane $T \in \mathcal{Q}'$ can be seen as a surface S_1 in a 4-dimensional space that has been projected on the plane. S_1 can also be projected on $\widetilde{\mathbb{Z}}^3$ ($\widetilde{\mathbb{Z}}^3$ is the space $\mathbb{Z}^3$ where the diagonals of the faces of the cubes have been added), and one obtains a surface S_2. When projected on the plane, S_2 is the lozenge-with-diagonals tiling $L_d(T)$.

2.3 Elementary operations

Consider a simply connected sub-graph L^1 of a lozenge tiling of the plane $L \in \mathcal{L}$. Denote by L^1_d the graph obtained by adding the diagonals of the lozenges. Then using the bijection between the first height function and quadri-tilings of L^1_d we obtain, in exactly the same way as Elkies, Kuperberg, Larsen, Propp [3] have for domino tilings, the following lemma:

Lemma 2.4 *Every quadri-tiling of L^1_d can be transformed into any other by a finite sequence of the following operations, (in brackets is the number of possible orientations for the graph corresponding to the operation):*

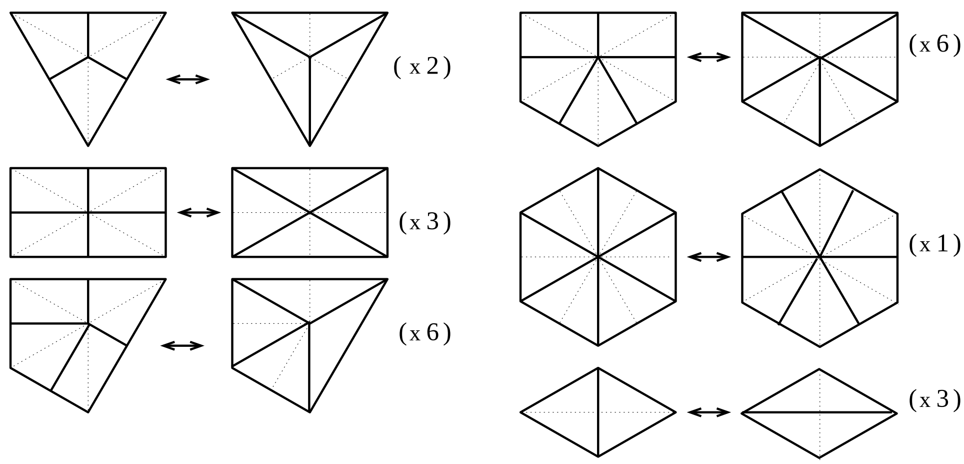

Figure 4: Quadri-tile operations.

Let us call **quadri-tile operations** the 21 operations described in the lemma.
Denote by ∂L^1 the boundary of L^1. Then every lozenge tiling of L^1 can be transformed into any other by a finite sequence of **lozenge operations**:

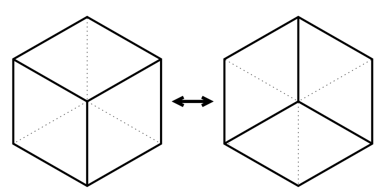

Figure 5: Lozenge operations.

Note that every lozenge-with-diagonals tiling of ∂L^1 is quadri-tileable with quadri-tiles obtained by cutting in two every lozenge along one of its diagonals. Moreover, when one performs a lozenge operation on such a tiling, one still obtains a quadri-tiling of ∂L^1. Let us call **elementary operations** the quadri-tile operations and the lozenge operations performed on quadri-tilings as described above. Then we have:

Lemma 2.5 *Every quadri-tiling of ∂L^1 can be transformed into any other by a finite sequence of elementary operations.*

Proof: This results from lemma 2.4, and the above observation. $\square$

3 Explicit Gibbs measure μ^R

Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, and let R_d be the corresponding rhombus-with-diagonals tiling. For a specific weight function on the edges of ${R_d}^*$ called the "critical" weight function, we construct a natural explicit Gibbs measure μ^R on $\mathcal{M}({R_d}^*)$.

3.1 Critical weight function

We follow [7] for the definition of the critical weight function. The critical weight function is defined on the edges of graphs satisfying a geometric condition called **isoradiality**: all faces of an isoradial graph are inscribable in a circle, and all circumcircles have the same radius (equal to one since we have chosen side-length two rhombi in $\mathcal{R}$). If $R \in \mathcal{R}$ is a rhombus tiling of the plane, then R_d is an isoradial graph. Note that the circumcenter of a face (right triangle) of the graph R_d is at the mid-point of the hypotenuse-boundary-edge of the face. Let us consider the embedding of the dual graph ${R_d}^*$ (the same notation is used for the one-skeleton of a graph and its embedding) where the dual vertices are the circumcenters of the corresponding faces. Then ${R_d}^*$ is an isoradial graph and the circumcenters of the faces are the vertices of R_d. To each edge e of ${R_d}^*$, we associate a unit-side length rhombus $R(e)$ whose vertices are the vertices of e and the vertices of its dual edge. Let $\widetilde{R} = \cup_{e \in {R_d}^*} R(e)$. Note that the dual edges corresponding to the boundary edges of the rhombi of R_d have length zero, and that the rhombi associated to these edges are degenerated.
For each edge e of ${R_d}^*$, define $\nu(e) = 2\sin\theta$, where 2θ is the angle of the rhombus $R(e)$ at the vertex it has in common with e. Note that $\nu(e)$ is the length of e^*, the dual edge of e. The function ν is called the **critical weight function**.

3.2 The Dirac and inverse Dirac operator

Results in this section are due to Kenyon [7], see also Mercat [13].
Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, then by lemma 2.2, ${R_d}^*$ is a bipartite graph. Recall B (resp. W) denotes the set of black (resp. white) vertices of ${R_d}^*$. Let ν be the critical weight function on the edges of ${R_d}^*$. The Hermitian matrix K indexed by the vertices of ${R_d}^*$ is defined as follows. If v_1 and v_2 are not adjacent $K(v_1, v_2) = 0$. If $w \in W$ and $b \in B$ are adjacent vertices, then $K(w,b) = \overline{K(b,w)}$ is the complex number of modulus $\nu(wb)$ and direction pointing from w to b. If w and b have the same image in the plane, then $|K(w,b)| = 2$, and the direction of $K(w,b)$ is that which is perpendicular to the corresponding dual edge, and has sign determined by the local orientation. The matrix K, also called **Kasteleyn matrix**, defines the **Dirac operator** $K : \mathbb{C}^{|{R_d}^*|} \to \mathbb{C}^{|{R_d}^*|}$, by

$$(Kf)(v) = \sum_{u \in {R_d}^*} K(v,u) f(u).$$

For the isoradial bipartite graph ${R_d}^*$, K^{-1} is defined to be the unique operator satisfying:

1. $KK^{-1} = \mathrm{Id}$,
2. $K^{-1}(b,w) \to 0$, when $|b - w| \to \infty$.

Let w be a white vertex of ${R_d}^*$. For every other vertex v, define a rational function $f_v(z)$ as follows. Let $w = v_0, v_1, v_2, \ldots, v_k = v$ be an edge-path of $\widetilde{R}$, from w to v. Each edge $v_j v_{j+1}$ has exactly one vertex of ${R_d}^*$ (the other is a vertex of R_d). Direct the edge away from this vertex if it is white, and towards this vertex if it is black. Let $e^{i\alpha_j}$ be the corresponding vector in $\widetilde{R}$ (which may point contrary to the direction of the path). f_v is defined inductively along the path, starting from

$$f_{v_0}(z) = 1.$$

If the edge leads away from a white vertex, or towards a black vertex, then

$$f_{v_{j+1}}(z) = \frac{f_{v_j}(z)}{z - e^{i\alpha_j}},$$

else, if it leads towards a white vertex, or away from a black vertex, then

$$f_{v_{j+1}}(z) = f_{v_j}(z).(z - e^{i\alpha_j}).$$

The function $f_v(z)$ is well defined (i.e. independent of the edge-path in $\widetilde{R}$ from w to v) because the multipliers for a path around a rhombus of $\widetilde{R}$ come out to 1. For a black vertex b the value $K^{-1}(b,w)$ will be the sum over the poles of $f_b(z)$ of the residue of f_b times the angle of z at the pole. However, there is an ambiguity in the choice of angle, which is only defined up to a multiple of 2π. To make this definition precise, angles are assigned to the poles of $f_b(z)$. Working on the branched cover of the plane, branched over w, so that for each black vertex b in this cover, a real angle θ_0 is assigned to the complex vector $b-w$, which increases by 2π when b winds once around w. In the branched cover of the plane, a real angle in $(\theta_0-\pi,\theta_0+\pi)$ can be assigned to each pole of f_b.

Theorem 3.1 [7] *There exists a unique K^{-1} satisfying the above two properties, and K^{-1} is given by:*

$$K^{-1}(b,w) = \frac{1}{4\pi^2 i}\int_C f_b(z)\log z\, dz,$$

where C is a closed contour surrounding cclw the part of the circle $\{e^{i\theta}|\theta\in[\theta_0-\pi+\varepsilon,\theta_0+\pi-\varepsilon]\}$, which contains all the poles of f_b, and with the origin in its exterior.

3.3 Explicit Gibbs measure μ^R

Let $R\in\mathcal{R}$ be a rhombus tiling of the plane, and R_d be the corresponding rhombus-with-diagonals tiling. If $\{e_1=w_1b_1,\dots,e_k=w_kb_k\}$ is a subset of edges of ${R_d}^*$, define the **cylinder set** E of ${R_d}^*$ as follows:

$$E=\{M\in\mathcal{M}({R_d}^*)\,|\,M \text{ contains the edges } e_1,\dots,e_k\}.$$

Let $\mathcal{A}$ be the field consisting of the empty set and the finite disjoint unions $\cup_{i=1}^m E^i$ of cylinders E^i. Denote by $\sigma(\mathcal{A})$ the σ-field generated by $\mathcal{A}$.

Theorem 3.2 *There exists a unique probability measure μ^R on $(\mathcal{M}({R_d}^*),\sigma(\mathcal{A}))$ that satisfies*

$$\mu^R(E) = \left(\prod_{i=1}^k K(w_i,b_i)\right)\det_{1\le i,\, j\le k}\left(K^{-1}(b_i,w_j)\right), \tag{3}$$

where K is the Kasteleyn matrix indexed by the vertices of ${R_d}^*$. Moreover μ^R is a Gibbs measure.

Remark 3.3

1. Let R_d^1 be a simply connected sub-graph of R_d, and let K^1 be the sub-matrix of K indexed by the vertices of R_d^{1*}. Then by [7], the probability that a subset of edges $\{w_1b_1,\dots,w_kb_k\}$ of R_d^{1*} occurs in a dimer configuration chosen with respect to the Boltzmann measure μ^{R_1}, is given by

$$\left(\prod_{i=1}^k K^1(w_i,b_i)\right)\det_{1\le i,\, j\le k}\left((K^1)^{-1}(b_i,w_j)\right).$$

   In that respect, our definition of the measure μ^R is a natural extension of the Boltzmann measure to $\mathcal{M}({R_d}^*)$.

2. The proof of theorem 3.2 uses propositions 3.4 and 3.8. Proposition 3.4 is a geometric property of rhombus tilings, it is stated and proved in section 3.4. Proposition 3.8 concerns convergence of the Boltzmann measure on some appropriate torus to (3), it is stated and proved in section 3.5. Theorem 3.2 is proved in section 3.6.

3.4 Geometric property of rhombus tilings

Proposition 3.4 *Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, then any finite simply connected sub-graph P of R can be embedded in a periodic rhombus tiling $\bar{R}$ of the plane.*

Proof: This proposition is a direct consequence of lemmas 3.5, 3.6, 3.7 below. □

The notion of **train-track** has been introduced by Mercat [12], see also Kenyon and Schlenker [7, 10]. A train-track of a rhombus tiling is a path of rhombi (each rhombus being adjacent along an edge to the previous rhombus) which does not turn: on entering a rhombus, it exits across the opposite edge. Train-tracks are assumed to be maximal in the sense that they extend in both directions as far as possible. Thus train-tracks of rhombus tilings of the plane are bi-infinite. Each rhombus in a train-track has an edge parallel to a fixed unit vector e, called the **transversal direction** of the train-track. Let us denote by t_e the train-track of transversal direction e. In an oriented train-track (i.e. the edges of the two parallel boundary paths of the train-track have the same given orientation), we choose the direction of e so that when the train-track runs in the direction given by the orientation, e points from the right to the left. The vector e is called the **oriented transversal direction** of the oriented train-track. A train-track cannot cross itself, and two different train-tracks can cross at most once. A finite simply connected sub-graph P of a rhombus tiling of the plane $R \in \mathcal{R}$ is **train-track-convex**, if every train-track of R that intersects P crosses the boundary of P twice exactly.

Lemma 3.5 *Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, then any finite simply connected sub-graph P of R can be completed by a finite number of rhombi of R in order to become train-track-convex.*

Proof: Let $e_1, \dots, e_m$ be the boundary edges of P. Every rhombus of P belongs to two train-tracks of R, each of which can be continued in both directions up to the boundary of P. In both directions the intersection of each of the train-tracks and the boundary of P is an edge parallel to the transversal direction of the train-track. Thus, to take into account all train-tracks of R that intersect P, it suffices to consider for every i the train-tracks t_{e_i} associated to the boundary edges of P. Consider the following algorithm (see figure 6).

Set $Q_1 = P$.
For $i = 1, \dots, m$, do the following:
Consider the train-track t_{e_i}, and let $2n_i$ be the number of times t_{e_i} intersects the boundary of Q_i.

- If $n_i > 1$: there are $n_i - 1$ portions of t_{e_i} that are outside of Q_i, denote them by ${t_{e_i}}^1, \dots, {t_{e_i}}^{n_i-1}$. Then, since Q_i is simply connected, for every j, $R \setminus (Q_i \cup {t_{e_i}}^j)$ is made of two disjoint sub-graphs of R, one of which is finite (it might be empty in the case where one of the two parallel boundary paths of ${t_{e_i}}^j$ is part of the boundary path of Q_i). Denote by ${g_{e_i}}^j$ the simply connected sub-graph of R made of the finite sub-graph of $R \setminus (Q_i \cup {t_{e_i}}^j)$ and of ${t_{e_i}}^j$. Denote by ${b_{e_i}}^j$ the portion of the boundary of Q_i which bounds ${g_{e_i}}^j$. Replace Q_i by $Q_{i+1} = Q_i \cup (\cup_{j=1}^{n_i-1} {g_{e_i}}^j)$. By this construction t_{e_i} intersects the boundary of Q_{i+1} exactly twice, and Q_{i+1} is simply connected.

- If $n_i = 1$: set $Q_{i+1} = Q_i$.

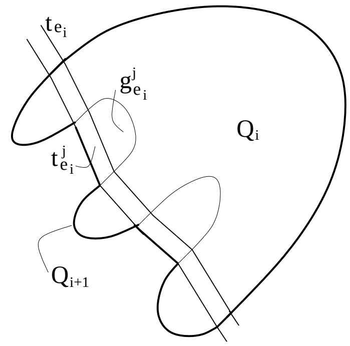

Figure 6: One step of the algorithm.

Let us show that at every step the train-tracks of R that intersect Q_i and Q_{i+1} are the same. By construction, boundary edges of Q_{i+1} are boundary edges of Q_i and of ${t_{e_i}}^j$, for every j. Let f be an edge on the boundary of Q_{i+1}, but not of Q_i, that is f is on the boundary of ${t_{e_i}}^j$ for some j, thus t_f crosses ${g_{e_i}}^j$. Since two train-tracks cross at most once, t_f has to intersect ${b_{e_i}}^j$, which means t_f also crosses Q_i. From this we also conclude that if a train-track intersects the boundary of Q_i twice, then it also intersects the boundary of Q_{i+1} twice.
Thus all train-tracks that intersect Q_{m+1} cross its boundary exactly twice, and Q_{m+1} contains P. $\square$

Lemma 3.6 *Let $R \in \mathcal{R}$ be a rhombus tiling of the plane. Then any finite simply connected train-track-convex sub-graph P of R can be completed by a finite number of rhombi in order to become a convex polygon Q, whose opposite boundary edges are parallel.*

Proof: Let $e_1, \ldots, e_m$ be the boundary edges of P oriented cclw. Since P is train-track-convex, the train-tracks $t_{e_1}, \ldots, t_{e_m}$ intersect the boundary of P twice, so that there are pairs of parallel boundary edges. Let us assume that the transversal directions of the train-tracks are all distinct (if this is not the case, one can always perturb the graph a little so that it happens). Let us also denote by $t_{e_1}, \ldots, t_{e_m}$ the portions of the bi-infinite train-tracks of R in P. In what follows, indices will be denoted cyclically, that is $e_j = e_{m+j}$. Write x_j (resp. y_j) for the initial (resp. end) vertex of an edge e_j.
Let e_i, e_{i+1} be two adjacent boundary edges of P. Consider the translate e^t_{i+1} of e_{i+1} so that the initial vertex of e^t_{i+1} is adjacent to the initial vertex of e_i. Then we define the **turning angle from** e_i **to** e_{i+1} (also called exterior angle) to be the angle $\widehat{e_i e^t_{i+1}}$, and we denote it by $\theta_{e_i, e_{i+1}}$. If e_i, e_j are two boundary edges, then the **turning angle from** e_i **to** e_j is defined by $\sum_{\alpha=i}^{j-1} \theta_{e_\alpha, e_{\alpha+1}}$, and is denoted by θ_{e_i, e_j}.

Properties

1. $\sum_{\alpha=1}^{m} \theta_{e_\alpha, e_{\alpha+1}} = 2\pi$.
2. If e_i, e_j are two boundary edges, and if $\gamma = \{f_1, \ldots, f_n\}$ is an oriented edge-path in P from y_i to x_j, then $\theta_{e_i, e_j} = \theta_{e_i, f_1} + \sum_{\alpha=1}^{n-1} \theta_{f_\alpha, f_{\alpha+1}} + \theta_{f_n, e_j}$.

3. If e_i is a boundary edge of P, and e_k is the second boundary edge at which t_{e_i} intersects the boundary of P, then $\theta_{e_i,e_k} = \pi$. Thus e_k and e_i are oriented in the opposite direction, and we denote e_k by ${e_i}^{-1}$.

4. P is convex, if and only if every train-track of P crosses every other train-track of P.

We first end the proof of lemma 3.6, and then prove properties 1. to 4.
Note that properties 1. and 2. are true for any finite simply connected sub-graph of R.
The number of train-tracks intersecting P is $n = m/2$. So that if every train-track crosses every other train-track, the total number of crossings is $n(n-1)/2$. Consider the following algorithm (see figure 7 for an example).

Set $Q_1 = P$, $n_1 =$ the number of train-tracks that cross in Q_1.
For $i = 1, 2, \ldots$ do the following:

- If $n_i = n(n-1)/2$: then by property 4, Q_i is convex.
- If $n_i < n(n-1)/2$: then by property 4, $\theta_{e_{j_i},e_{j_i+1}} < 0$ for some $j_i \in \{1, \ldots, m\}$. Add the rhombus l_{j_i} of parallel directions e_{j_i}, e_{j_i+1} along the boundary of Q_i. Set $Q_{i+1} = Q_i \cup l_{j_i}$, and rename the boundary edges $e_1, \ldots, e_m$ in cclw order. Then the number of train-tracks that cross in Q_{i+1} is $n_i + 1$, set $n_{i+1} = n_i + 1$. Note that if property 4. is true for Q_i, it stays true for Q_{i+1}, and note that the same train-tracks intersect Q_i and Q_{i+1}.

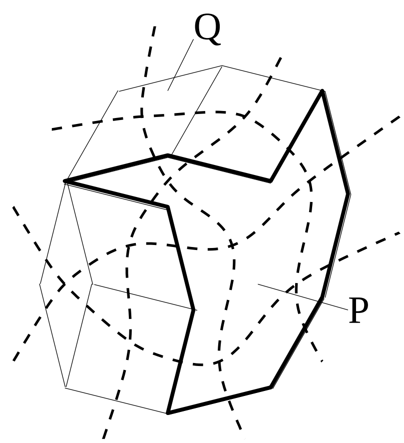

Figure 7: Example of application of the algorithm.

For the algorithm to be able to add the rhombus l_{j_i} at every step, we need to check that:

$$\forall\, i,\ \theta_{e_{j_i-1},e_{j_i+1}} > -\pi,\ \text{and}\ \theta_{e_{j_i},e_{j_i+2}} > -\pi. \tag{4}$$

Assume we have proved that for any finite simply connected train-track-convex sub-graph P of R we have:

$$\forall\, i, j,\ \theta_{e_i,e_j} > -\pi. \tag{5}$$

Then properties 1. and 2. imply that if (5) is true for Q_i it stays true for Q_{i+1}, moreover (5) implies (4). So let us prove (5) by induction on the number of rhombi contained in P. If P is a rhombus, then (5) is clear. Now assume P is made of k rhombi. Consider the train-tracks in P adjacent to the boundary (every boundary edge e of P belongs to a rhombus of P which has parallel directions

e and f; for every boundary edge e, the train-track of transversal direction f is the train-track adjacent to the boundary). Denote the train-tracks adjacent to the boundary by $t_1, \ldots, t_p$ in cclw order, and write f_β for the oriented transversal direction of t_β (when the boundary edge-path of P is oriented cclw). Consider two adjacent boundary edges e_i, e_{i+1} of P that don't belong to the same boundary train-track. That is e_i belongs to t_β, and e_{i+1} to $t_{\beta+1}$. Then either $\widehat{f_\beta f_{\beta+1}} < 0$ or $\widehat{f_\beta f_{\beta+1}} > 0$, in the second case t_β and $t_{\beta+1}$ cross and their intersection is a rhombus l_β of P. l_β has boundary edges e_i, e_{i+1}, and $f_{\beta+1} = e_i{}^{-1}, f_\beta{}^{-1} = e_{i+1}{}^{-1}$. Now property 1. implies that $\sum_{\beta=1}^{p-1} \widehat{f_\beta f_{\beta+1}} = 2\pi$, so that there always exists β_0 such that $\widehat{f_{\beta_0} f_{\beta_0+1}} > 0$. Removing l_{β_0} from P and using the assumption that P is train-track convex, we obtain a graph P' made of $k-1$ rhombi which is train-track-convex. By induction, $\theta_{e,f} > -\pi$ for every boundary edge of P', and using property 2, we conclude that this stays true for P.

Denote by Q the convex polygon obtained from P by the algorithm, and assume that opposite boundary edges are not parallel. Then there are indices i and j such that e_i comes before e_j, and e_j^{-1} comes before e_i^{-1}. This implies that $\theta_{e_i,e_j} = -\theta_{e_j{}^{-1},e_i{}^{-1}}$, so that one of the two angles is negative, which means Q can not be convex. Thus we have a contradiction, and we conclude that opposite boundary edges of Q are parallel.

Proof of properties 1. to 4.

1. and 2. are straightforward.

3. When computing θ_{e_i,e_k} along the boundary edge-path of t_{e_i} we obtain π, so by property 2. we deduce that $\theta_{e_i,e_k} = \pi$ in P.

4. P is convex if and only if, for every i, $\theta_{e_i,e_{i+1}} > 0$, which is equivalent to saying that, for every $i \neq j$, $\theta_{e_i,e_j} > 0$. Therefore property 4. is equivalent to proving that $\theta_{e_i,e_j} > 0$, for every $i \neq j$, if and only if every train-track of P crosses every other train-track of P.
   Assume there are two distinct train-tracks t_{e_l} and t_{e_k} that don't cross in P. Then, in cclw order around the boundary of P, we have either $e_l, e_k{}^{-1}, e_k, e_l{}^{-1}$, or $e_l, e_k, e_k{}^{-1}, e_l{}^{-1}$. It suffices to solve the second case, the first case being similar. By property 1, $\theta_{e_l,e_k} + \theta_{e_k,e_k{}^{-1}} + \theta_{e_k{}^{-1},e_l{}^{-1}} + \theta_{e_l{}^{-1},e_l} = 2\pi$. Moreover by property 3, $\theta_{e_k,e_k{}^{-1}} = \theta_{e_l{}^{-1},e_l} = \pi$, which implies $\theta_{e_l,e_k} = -\theta_{e_k{}^{-1},e_l{}^{-1}}$. Since all train-tracks have different transversal directions, either θ_{e_l,e_k} or $\theta_{e_k{}^{-1},e_l{}^{-1}}$ is negative.
   Now take two boundary edges e_i, e_j of P (with $i \neq j$, and $e_j \neq e_i{}^{-1}$), and assume the train-tracks t_{e_i}, t_{e_j} cross inside P. Then in cclw order around the boundary of P, we have either $e_i, e_j{}^{-1}, e_i{}^{-1}, e_j$, or $e_i, e_j, e_i{}^{-1}, e_j{}^{-1}$. It suffices to solve the second case since the first case can be deduced from the second one. The intersection of t_{e_i} and t_{e_j} is a rhombus l. Let $\tilde{e}_j^{-1}$ (resp. $\tilde{e}_i^{-1}$) be the boundary edge of l parallel and closest to e_j (resp. e_i), oriented in the opposite direction, then $\theta_{\tilde{e}_j^{-1},\tilde{e}_i^{-1}} < 0$. Let γ_j (resp. γ_i) be the boundary edge-path of t_{e_j} (resp. t_{e_i}) from y_j to $\tilde{x}_j$ (resp. from $\tilde{y}_i$ to x_i), and let Q be the sub-graph of R whose boundary is $e_i, e_{i+1}, \ldots, e_j, \gamma_j, \tilde{e}_j^{-1}, \tilde{e}_i^{-1}, \gamma_i$. Since t_{e_i} and t_{e_j} intersect the boundary of P twice, they also intersect the boundary of Q twice. Moreover t_{e_i} and t_{e_j} don't cross in Q, so that $\theta_{e_i,e_j} = -\theta_{\tilde{e}_j^{-1},\tilde{e}_i^{-1}} > 0$.

□

Lemma 3.7 *Any convex $2n$-gon Q whose opposite boundary edges are parallel and of the same length can be embedded in a periodic tiling of the plane by Q and rhombi.*

Proof: Let $e_1, \dots, e_n, {e_1}^{-1}, \dots, {e_n}^{-1}$ be the boundary edges of the polygon Q oriented cclw. If $n \leq 3$, then Q is either a rhombus or a hexagon, and it is straightforward that the plane can be tiled periodically with Q.
If $n > 4$, for $k = 1, \dots, n-3$, do the following (see figure 8): along e_{n-k} add the finite train-track $\tilde{t}_{e_{n-k}}$ of transversal direction e_{n-k}, going away from Q, whose boundary edges starting from the boundary of Q are:

$$e_1, \underbrace{e_2, e_1}, \underbrace{e_3, e_2, e_1}, \dots, \underbrace{e_{n-k-2}, \dots, e_1}.$$

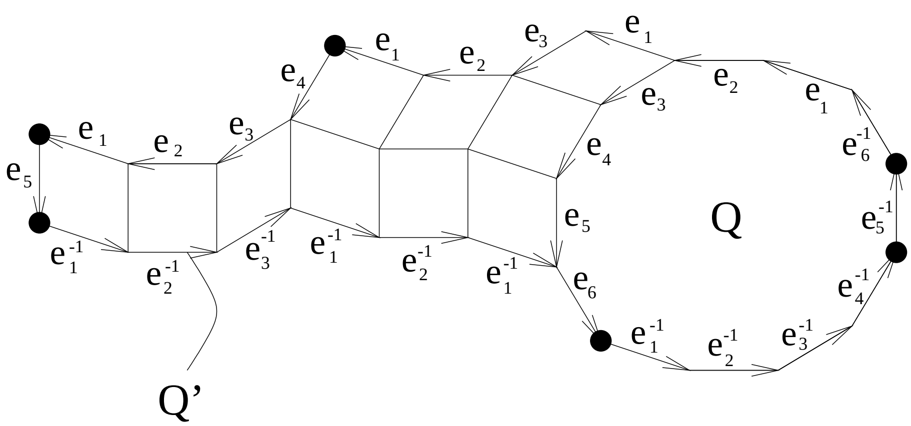

Figure 8: Fundamental domain of a periodic tiling of the plane by dodecagons and rhombi.

Since the polygon Q is convex, the rhombi that are added are well defined, moreover the intersection of $\tilde{t}_{e_i}$ and the boundary of Q is the edge e_i, and $\tilde{t}_{e_i}$ doesn't cross $\tilde{t}_{e_j}$ when $i \neq j$. So we obtain a new polygon Q' made of Q and rhombi, whose boundary edge-path is $\gamma_1, \dots, \gamma_6$, (when starting from the edge ${e_n}^{-1}$ of Q), where:

$$\gamma_1 = {e_n}^{-1}, e_1, \underbrace{e_2, e_1}, \underbrace{e_3, e_2, e_1}, \dots, \underbrace{e_{n-3}, \dots, e_1},$$

$$\gamma_2 = e_{n-2}, \dots, e_1,$$

$$\gamma_3 = e_{n-1},$$

$$\gamma_4 = \underbrace{{e_1}^{-1}, \dots, {e_{n-3}}^{-1}}, \dots, \underbrace{{e_1}^{-1}, {e_2}^{-1}}, {e_1}^{-1}, e_n,$$

$$\gamma_5 = {e_1}^{-1}, \dots, {e_{n-2}}^{-1},$$

$$\gamma_6 = {e_{n-1}}^{-1}.$$

Noting that $\gamma_4 = {\gamma_1}^{-1}, \gamma_5 = {\gamma_2}^{-1}, \gamma_6 = {\gamma_3}^{-1}$, and using the fact that the plane can be tiled periodically with hexagons which have parallel opposite boundary edges, we deduce that the plane can be tiled with Q', that is it can be tiled periodically by Q and rhombi. $\square$

3.5 Convergence of the Boltzmann measure on a torus

Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, and let R_d be the corresponding rhombus-with-diagonals tiling. Let $\{e_1 = w_1 b_1, \dots, e_k = w_k b_k\}$ be a subset of edges of ${R_d}^*$. Consider P a finite simply connected sub-graph of the graph R such that ${P_d}^*$ contains these edges. Then by proposition 3.4, there exists a periodic rhombus tiling of the plane $\bar{R}$ that contains P. The dual $\bar{R}_d^*$ of the rhombus-with-diagonals tiling $\bar{R}_d$ corresponding to $\bar{R}$ is bipartite and invariant under the translates

of a two dimensional lattice Λ of $\mathbb{R}^2$. Let us call horizontal and vertical the translations by two basis vectors of Λ. Each of these translations either preserves the colouring of the vertices of $\bar{R}_d^*$ or exchanges it from black to white, and from white to black. That is, if instead we take horizontal and vertical translations by twice the basis vectors of Λ, then the colouring of the vertices is preserved by translation. Denote by $\bar{R}_{d,n}$ the quotient $\bar{R}_d/n(2\Lambda), n \in \mathbb{N}$ embedded on the torus, then $\bar{R}_{d,n}^*$ is a bipartite graph on the torus.
Assume the critical weight function is associated to the edges of $\bar{R}_{d,n}^*$, and denote by $\mu_n^{\bar{R}}$ the Boltzmann measure on $\mathcal{M}(\bar{R}_{d,n}^*)$. Moreover define

$$p(e_1, \ldots, e_k) = \left(\prod_{i=1}^{k} K(w_i, b_i) \right) \det_{1 \le i,\, j \le k} \left(K^{-1}(b_i, w_j) \right),$$

where K is the Kasteleyn matrix indexed by the vertices of $R_d{}^*$. Let E_n be the set of dimer configurations of $\bar{R}_{d,n}^*$ that contain the edges $\{e_1, \ldots, e_k\}$, then

Proposition 3.8

$$\lim_{n \to \infty} \mu_n^{\bar{R}}(E_n) = p(e_1, \ldots, e_k).$$

Proof: Write $\bar{R}_d(i,j)$ for the i^{th} horizontal and j^{th} vertical translate of the fundamental domain of $\bar{R}_d$ in $\bar{R}_{d,n}$. Let K_1^n be the Kasteleyn matrix indexed by the vertices of the graph $\bar{R}_{d,n}^*$. Denote by K_2^n (resp. K_3^n) the matrix obtained from K_1^n by changing, for $i, j = 1, \ldots, n$, the sign of the entries corresponding to the edges from $\bar{R}_d(1,i)$ to $\bar{R}_d(n,j)$ (resp. from $\bar{R}_d(i,1)$ to $\bar{R}_d(j,n)$); and denote by K_4^n the matrix obtained from K_1^n by changing the sign of both of these entries. If we let B_n (resp. W_n) be the set of black (resp. white) vertices of $\bar{R}_{d,n}^*$, then for $l = 1, \ldots, 4$, K_l^n can be written as:

$$K_l^n = \begin{pmatrix} 0 & \widetilde{K_l^n} \\ \widetilde{K_l^n}^* & 0 \end{pmatrix},$$

where $\widetilde{K_l^n} : \mathbb{C}^{|W_n|} \to \mathbb{C}^{|B_n|}$, and $\widetilde{K_l^n}^*$ denotes the conjugate transpose of the matrix $\widetilde{K_l^n}$.

Theorem 3.9 [6, 7] *The dimer partition function Z_n of $\bar{R}_{d,n}^*$ is*

$$Z_n = \frac{1}{2} \left| -\det \widetilde{K_1^n} + \det \widetilde{K_2^n} + \det \widetilde{K_3^n} + \det \widetilde{K_4^n} \right|.$$

Theorem 3.10 [8]

$$\begin{aligned} \mu_n^{\bar{R}}(E_n) = \frac{1}{2} \Bigg(\prod_{i=1}^{k} K(w_i, b_i) \Bigg\{ -\frac{\det \widetilde{K_1^n}}{Z_n} \det_{1 \le i,j \le k} \left((K_1^n)^{-1}(b_i, w_j) \right) \\ + \sum_{l=2}^{4} \frac{\det \widetilde{K_l^n}}{Z_n} \det_{1 \le i,j \le k} \left((K_l^n)^{-1}(b_i, w_j) \right) \Bigg\} \Bigg). \end{aligned} \tag{6}$$

Theorem 3.11 [9] *For $l = 1, \ldots, 4$ and for every $b \in B_n$, $w \in W_n$*

$$\lim_{n \to \infty} (K_l^n)^{-1}(b, w) = \frac{1}{(2\pi)^2} \int_{S^1 \times S^1} \frac{Q_{b,w}(z,u) u^x z^y}{P(z,u)} \frac{dz}{z} \frac{du}{u}, \tag{7}$$

where $Q_{b,w}$ and P are polynomials, and x (resp. y) is the horizontal (resp. vertical) translation from the fundamental domain of b to the fundamental domain of w. Moreover the right hand side of (7) converges to 0 as $|b - w| \to \infty$.

Denote by B_P (resp. W_P) the set of black (resp. white) vertices of ${P_d}^*$, and let us prove that as a consequence of theorem 3.11 we have:

$$\lim_{n\to\infty}(K_l^n)^{-1}(b,w)=K^{-1}(b,w),\ \forall\, b\in B_P,\ w\in W_P.$$

Let $K_{\bar{R}}$ be the Dirac operator associated to the isoradial graph $\bar{R}_d^*$, and let ${K_{\bar{R}}}^{-1}$ be the unique inverse Dirac operator given by theorem 3.1. Let us write $B_{\bar{R}}$ (resp. $W_{\bar{R}}$) for the set of black (resp. white) vertices of $\bar{R}_d^*$. Then, by theorem 3.1 of [7], for $b\in B_{\bar{R}}$, $w\in W_{\bar{R}}$, ${K_{\bar{R}}}^{-1}(b,w)$ only depends on an edge-path from w to b, so that:

$$ {K_{\bar{R}}}^{-1}(b,w)=K^{-1}(b,w),\ \forall\, b\in B_P,\ w\in W_P. \tag{8}$$

Now take $w_0,w_0'\in W_{\bar{R}}$, and let b_1,b_2,b_3 be the black neighbours of w_0 (vertices of $\bar{R}_d^*$ are of degree 3), then if n is large enough we have (for $l=1,\dots,4$):

$$K_l^n(b_i,w_0)=K_{\bar{R}}(b_i,w_0),\ \forall\, i=1,\dots 3,$$

so that:

$$\sum_{b\in B_{\bar{R}}}K_l^n(w_0,b)(K_l^n)^{-1}(b,w_0')=\delta_{w_0w_0'}\Leftrightarrow\sum_{i=1}^{3}K(w_0,b_i)(K_l^n)^{-1}(b_i,w_0')=\delta_{w_0w_0'}. \tag{9}$$

Denote by $F(b,w)$ the right hand side of (7), then taking the limit as $n\to\infty$ in (9), we obtain:

$$\sum_{i=1}^{3}K_{\bar{R}}(w_0,b_i)F(b_i,w_0')=\delta_{w_0w_0'}.$$

This is true for all pairs of white vertices w_0,w_0' of $\bar{R}_d^*$. Moreover $\lim_{|b-w|\to\infty}F(b,w)=0$, so that by uniqueness of the inverse Dirac operator, we have:

$$F(b,w)={K_{\bar{R}}}^{-1}(b,w),\ \forall\, b\in B_{\bar{R}},\ w\in W_{\bar{R}}, \tag{10}$$

and using equation (8) we conclude that:

$$F(b,w)=K^{-1}(b,w),\ \forall\, b\in B_P,\ w\in W_P. \tag{11}$$

We end the proof of proposition 3.8 following an argument of [8]. The probability $\mu_n^{\bar{R}}(E_n)$ given in (6) is a weighted average of the four quantities $\det_{1\le i,j\le k}\left((K_l^n)^{-1}(b_i,w_j)\right)$, with weights $\pm\det\widetilde{K_l^n}/Z_n$. These weights are all in the interval $(-1,1)$ since $Z_n>|\det\widetilde{K_l^n}|$ for every $l=1,\dots,4$. Indeed, Z_n counts the weighted number of dimer configurations of $\bar{R}_{d,n}^*$, whereas $|\det\widetilde{K_l^n}|$ counts some configurations with negative sign. Since the weights sum to 1, the weighted average converges to the same value as each $\det_{1\le i,j\le k}\left((K_l^n)^{-1}(b_i,w_j)\right)$. So that, using equation (11), $\mu_n^{\bar{R}}(E_n)$ converges to $p(e_{t_1},\dots,e_{t_k})$ as $n\to\infty$. $\square$

3.6 Proof of theorem 3.2

Let us first recall the setting of theorem 3.2. We consider a rhombus tiling of the plane $R \in \mathcal{R}$, and we let R_d be the corresponding rhombus-with-diagonals tiling. The edges of ${R_d}^*$ form a countable set. For every $i \in \mathbb{N}$, define $f_i : \mathcal{M}({R_d}^*) \to \{0,1\}$ by

$$f_i(M) = \begin{cases} 1 & \text{if the edge } e_i \text{ belongs to } M, \\ 0 & \text{else.} \end{cases}$$

For each k-tuple $(s_1, \dots, s_k)$ of distinct elements of $\mathbb{N}$, a **cylinder A^k of rank k** is of the form

$$A^k = \{M \in \mathcal{M}({R_d}^*) | (f_{s_1}(M), \dots, f_{s_k}(M)) \in H,\ H \in \mathcal{B}\{0,1\}^k\},$$

where $\mathcal{B}\{0,1\}^k$ is the Borel σ-field of $\{0,1\}^k$. Denote by $\mathcal{A}'$ the set of cylinders of all ranks, then $\mathcal{A}'$ is a field. Moreover $\mathcal{A}' = \mathcal{A}$ (recall $\mathcal{A}$ is the field of disjoint unions of cylinders E^i).

Fix $k \in \mathbb{N}$. Let $H \in \mathcal{B}\{0,1\}^k$, and let A^k be the corresponding cylinder. Then $A^k = \cup_{i=1}^m E^i$, where E^i is the set of dimer configurations of ${R_d}^*$ containing the edges $\{e^i_{t_1}, \dots, e^i_{t_{l_i}}\}$. Define

$$\mu^R_{s_1,\dots,s_k}(H) = \sum_{i=1}^m p(e^i_{t_1}, \dots, e^i_{t_{l_i}}).$$

Let ${P_d}^*$ be a finite simply connected sub-graph of ${R_d}^*$ containing the edges $\{e^i_{t_1}, \dots, e^i_{t_{l_i}}\}$ for every i. Let $\bar{R}$ be a periodic rhombus tiling of the plane that contains P. Recall $\mu^{\bar{R}}_n$ is the Boltzmann measure on $\mathcal{M}(\bar{R}^*_{d,n})$. Define

$$\begin{aligned} A^k_n &= \{M \in \mathcal{M}(\bar{R}^*_{d,n}) \,|\, (f_{s_1}(M), \dots, f_{s_k}(M)) \in H,\ H \in \mathcal{B}\{0,1\}^k\}, \\ E^i_n &= \{M \in \mathcal{M}(\bar{R}^*_{d,n}) \,|\, M \text{ contains the edges } e_{t^i_1}, \dots, e^i_{t_{l_i}}\}. \end{aligned}$$

Then $A^k_n = \cup_{i=1}^m E^i_n$, and we have $\mu^{\bar{R}}_n(A^k_n) = \sum_{i=1}^m \mu^{\bar{R}}_n(E^i_n)$.
Moreover by proposition 3.8, $\lim_{n\to\infty} \mu^{\bar{R}}_n(E^i_n) = p(e_{t_1}, \dots, e_{t_{l_i}})$, so that

$$\lim_{n\to\infty} \mu^{\bar{R}}_n(A^k_n) = \mu^R_{s_1,\dots,s_k}(H).$$

From this we deduce that for every k, and for every k-tuple $(s_1, \dots, s_k)$, $\mu^R_{s_1,\dots,s_k}$ is a probability measure on $\mathcal{B}\{0,1\}^k$, that satisfies Kolmogorov's two consistency conditions. Applying Kolmogorov's extension theorem, we obtain theorem 3.2.

Using the fact that the measure μ^R of a cylinder set is the limit of the Boltzmann measure, we deduce that the measure μ^R is a Gibbs measure in the sense given in the introduction.

□

4 Explicit measure μ

Recall $\mathcal{M}' = \cup_{L\in\mathcal{L}} \mathcal{M}({L_d}^*)$ is the set of all dimer configurations of duals of lozenge-with-diagonals tilings of the plane. In the following paragraphs we construct an explicit measure μ on $\mathcal{M}'$.

Quadri-tilings are defined up to isometry, which means we consider lozenge tilings of the plane that correspond to dimer configurations of the honeycomb lattice H. Consider the graphs $\Gamma = \cup_{L\in\mathcal{L}} L_d$ and $\Gamma^* = \cup_{L\in\mathcal{L}} {L_d}^*$, Γ and Γ^* are not planar, but the edges of Γ^* form a countable set. In the isoradial embedding of the graph ${L_d}^*$ some edges have length zero, and so the quadri-tile dual of an edge of Γ^* is not uniquely defined. Thus we choose an embedding of the graphs ${L_d}^*$ which has the property that each edge of Γ^* corresponds to a unique quadri-tile. Let e_j be an edge of Γ^*, the dual quadri-tile of e_j is made of two right triangles that belong to either one or two lozenges, which we call the **lozenge(s) associated to the edge** e_j.

Let $\{e_1 = w_1 b_1, \ldots, e_k = w_k b_k\}$ be a subset of edges of Γ^* where the lozenges associated to the edges form a connected path (that is each lozenge is adjacent along an edge to the previous one). Define the **cylinder set** $\mathcal{E}$ of $\mathcal{M}'$ by

$$\mathcal{E} = \{M \in \mathcal{M}' \mid M \text{ contains the edges } e_1, \ldots, e_k\},$$

and denote by $\sigma(\mathcal{B})$ the σ-field generated by the finite disjoint unions $\cup_{i=1}^m \mathcal{E}^i$ of cylinder sets $\mathcal{E}^i$ of $\mathcal{M}'$.

Let $\sigma(\mathcal{C})$ be the σ-field generated by the finite disjoint unions of cylinder sets of H. Assume we have a measure ν on $(\mathcal{M}(H), \sigma(\mathcal{C}))$. An example of such a measure is given in [8]: it is defined as the limit of the Boltzmann measure on the torus, corresponding to weights 1 on the edges of H (these are the critical weights for the honeycomb lattice). Denote by $\{f_1, \ldots, f_l\}$ the edges of H dual of the lozenges associated to the edges $\{e_1, \ldots, e_k\}$, and let F be the cylinder set of H corresponding to the edges $\{f_1, \ldots, f_l\}$.

Theorem 4.1 *There exists a unique probability measure μ on $(\mathcal{M}', \sigma(\mathcal{B}))$ that satisfies*

$$\mu(\mathcal{E}) = \nu(F) \mu^L(E), \tag{12}$$

where $L \in \mathcal{L}$ is a lozenge tiling of the plane that contains the lozenges associated to the edges $e_1, \ldots, e_k$, and E is the cylinder set of ${L_d}^*$ corresponding to the edges $e_1, \ldots, e_k$.

Proof: Expression (12) is well defined. Indeed, by definition, the lozenges associated to the edges $e_1, \ldots, e_k$ form a connected path, say γ. Let $L \in \mathcal{L}$ be a lozenge tiling that contains γ, and denote by K_L the Dirac operator indexed by the vertices of the graph ${L_d}^*$. Then $K_L^{-1}(b_i, w_j)$ is independent of the lozenge tiling L, indeed $K_L^{-1}(b_i, w_j)$ only depends on an edge-path of $\tilde{R}$ (the set of rhombi associated to the edges of ${L_d}^*$) from w_j to b_i, and since L contains γ which is connected, we can choose the edge-path to be the same for any such lozenge tiling L. We then use the fact that ν and μ^L are probability measures to prove the conditions of Kolmogorov's extension theorem. $\square$

5 Asymptotics of the inverse Dirac operator and of the measures μ^R and μ

Let $R \in \mathcal{R}$ be a rhombus tiling of the plane, and let R_d be the corresponding rhombus-with-diagonals tiling. Consider the inverse Dirac operator K^{-1} indexed by the vertices of ${R_d}^*$. We establish that asymptotically (as $|b - w| \to \infty$), $K^{-1}(b, w)$ only depends on the rhombi to which the vertices b and w belong, and else is independent of the graph ${R_d}^*$. We conclude with asymptotic properties of the measures μ^R and μ.

5.1 Asymptotics of the inverse Dirac operator

Refer to figure 9 for the following notations. Let l_1', l_2' be two disjoint side-length two rhombi in the plane, and let l_1, l_2 be the corresponding rhombi with diagonals. Assume l_1 and l_2 have a fixed black and white bipartite colouring of their faces. Let r_1 and r_2 be the dual graphs of l_1 and l_2 (r_1 and r_2 are rectangles), with the corresponding bipartite colouring of the vertices. Let w be a white vertex of r_1, and b a black vertex of r_2, then w (resp. b) belongs to a boundary edge e_1 of l_1 (resp. e_2 of l_2). By lemma 2.2, to the bipartite colouring of the faces of l_1 and l_2, there corresponds a bipartite colouring of the vertices of l_1' and l_2'. Let x_1 (resp. x_2) be the black vertex of the edge e_1 (resp. e_2). Orient the edge wx_1 from w to x_1, and let $e^{i\theta_1}$ be the corresponding vector. Orient the edge x_2b from x_2 to b, and let $e^{i\theta_2}$ be the corresponding vector. Assume l_1' and l_2' belong to a rhombus tiling of the plane $R \in \mathcal{R}$. Moreover, suppose that the bipartite colouring of the vertices of ${R_d}^*$ is compatible with the bipartite colouring of the vertices of r_1 and r_2.

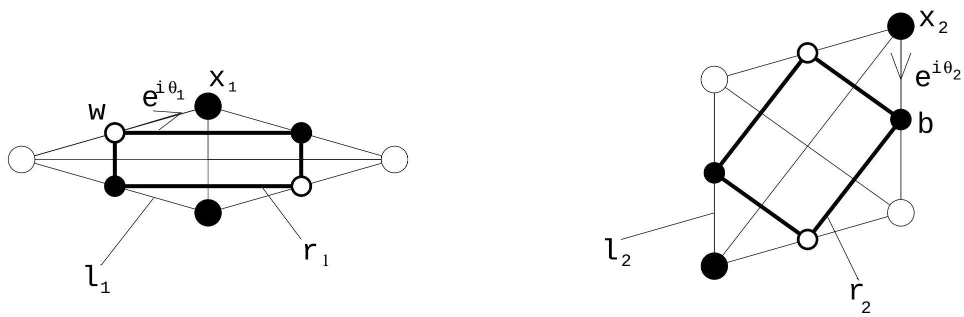

Figure 9: Rhombi with diagonals l_1, l_2 and their dual graphs r_1, r_2.

Then we have the following asymptotics for the inverse Dirac operator K^{-1} indexed by the vertices of ${R_d}^*$.

Theorem 5.1

$$K^{-1}(b,w) = \frac{1}{2\pi}\left(\frac{1}{b-w} + \frac{e^{-i(\theta_1+\theta 2)}}{\bar{b}-\bar{w}}\right) + \frac{1}{2\pi}\left(\frac{e^{2i\theta_1}+e^{2i\theta_2}}{(b-w)^3} + \frac{e^{-i(3\theta_1+\theta_2)}+e^{-i(\theta_1+3\theta_2)}}{(\bar{b}-\bar{w})^3}\right) + O\left(\frac{1}{|b-w|^3}\right), \tag{13}$$

where θ_1 and θ_2 are defined above.

Proof: Let us define an edge-path from w to b in $\widetilde{R}$ (the set of rhombi associated to the edges of ${R_d}^*$). Consider the bipartite colouring of the vertices of R (given by lemma 2.2) associated to the bipartite colouring of the vertices of ${R_d}^*$. We define the graph N as follows. Vertices of N are black vertices of R, and two vertices of N are connected by an edge if they belong to the same rhombus in R. N is connected because R is. Each face of N is inscribable in a circle of radius two. The circumcenter of a face of N is the intersection of the rhombi in R, to which the edges on the boundary cycle of the face belong. Thus the circumcenter is in the closure of the face, and so faces of N are convex. Note that the vertices x_1 and x_2 are vertices of the graph N.
Denote by (x,y) the line segment from a vertex x to a vertex y of N. An edge uv of N is called a **forward-edge** for the segment (x,y) if $< v-u, y-x > \geq 0$. An edge-path $v_1,\dots,v_k$ of N is called a **forward-path** for the segment (x,y), if all the edges v_iv_{i+1} are forward-edges for (x,y). Similarly to what has been done in [7], let us define a forward-path of N for the segment (x_1,x_2), from x_1 to x_2 (see figure 10). Let $F_1,\dots,F_l$ be the faces of N whose interior intersect (x_1,x_2) (if some edge of N lies exactly on (x_1,x_2), perturb the segment (x_1,x_2) slightly, using instead a segment $(x_1+\varepsilon_1, x_2+\varepsilon_2)$ for two generic infinitesimal translations $(\varepsilon_1,\varepsilon_2)$). Note that the number

of such faces is finite because the rhombus tiling of the plane R has only finitely many different rhombi. Then for $j = 1, \ldots, l-1$, $F_j \cap F_{j+1}$ is an edge e_{j+1} of N crossing (x_1, x_2). Set $v_1 = x_1$, $v_l = x_2$, and for $j = 1, \ldots, l-2$, define v_{j+1} to be the vertex of e_{j+1} such that the edge e_{j+1} oriented towards v_{j+1} is a forward-edge for (x_1, x_2). Then, for $j = 1, \ldots, l-1$, the vertices v_j and v_{j+1} belong to the face F_j. Take an edge-path from v_j to v_{j+1} on the boundary cycle of F_j, such that it is a forward-path for (x_1, x_2). Such a path exists because faces of N are convex. Thus, we have built a forward-path of N for (x_1, x_2), from x_1 to x_2. Denote by $u_1 = x_1, u_2, \ldots, u_{k-1}, u_k = x_2$ the vertices of this path.

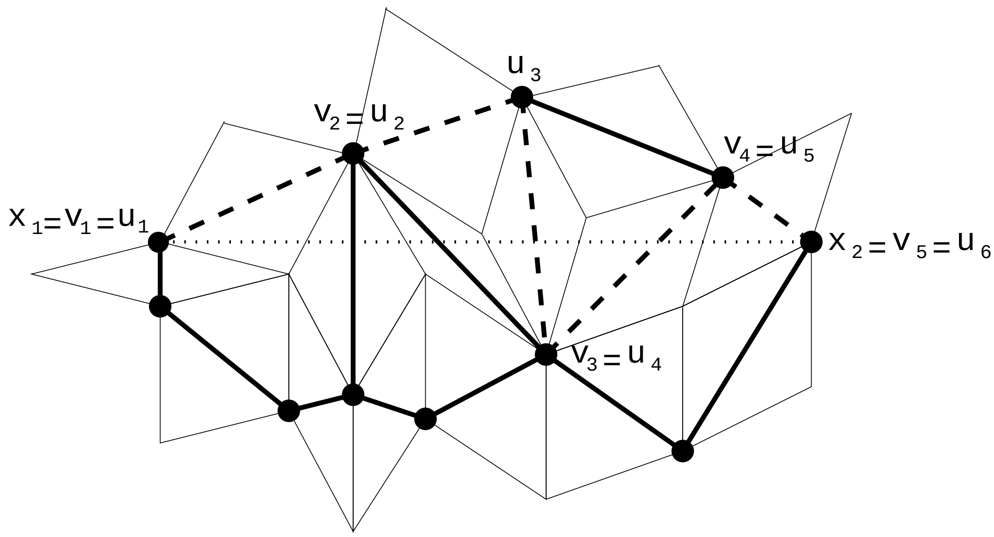

Figure 10: forward-path from x_1 to x_2 for the segment (x_1, x_2).

Let us now define an edge-path of $\widetilde{R}$ from w to b. Note that the edges wx_1 and x_2b are edges of $\widetilde{R}$. For $j = 1, \ldots, k-1$, define the following edge-path of $\widetilde{R}$ from u_j to u_{j+1} (see figure 11). Remember that $u_j u_{j+1}$ is the diagonal of a rhombus of R, say $\tilde{l}'_j$. Let $\tilde{r}_j$ be the dual graph of $\tilde{l}_j$. Let u_j^1 be the black vertex in $\tilde{r}_j$ adjacent to u_j, let u_j^2 be the crossing of the diagonals of $\tilde{l}_j$, and let u_j^3 be the white vertex in $\tilde{r}_j$ adjacent to u_{j+1}. Then the path $u_j, u_j^1, u_j^2, u_j^3, u_{j+1}$ is an edge-path of $\widetilde{R}$. Thus $w, x_1 = u_1, u_1^1, u_1^2, u_1^3, u_2, \ldots, u_{k-1}, u_{k-1}^1, u_{k-1}^2, u_{k-1}^3, u_k = x_2, b$ is an edge-path of $\widetilde{R}$, from w to b. Orient the edges in the path towards the black vertices of ${R_d}^*$, and away from the white vertices of ${R_d}^*$.

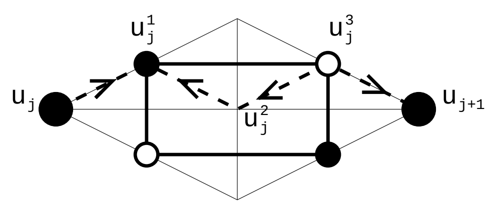

Figure 11: Edge-path of $\widetilde{R}$ from u_j to u_{j+1}.

Let $e^{i\beta_j^1}$, $e^{i\beta_j^2}$, $e^{i\alpha_j^1}$, $e^{i\alpha_j^2}$ be the vectors corresponding respectively to the edges $u_j u_j^1$, $u_j^3 u_{j+1}$, $u_j^2 u_j^1$, $u_j^3 u_j^2$. Without loss of generality, suppose that $x_2 - x_1$ is real and positive. Then for $j = 1, \ldots, k-1$, and $l = 1, 2$, we have:

$$\cos\beta_j^l - \cos\alpha_j^l = \frac{< u_{j+1} - u_j, x_2 - x_1 >}{2|x_2 - x_1|}.$$

Since $u_1, \ldots, u_k$ is a forward-path for (x_1, x_2), this quantity is positive, thus $\cos\beta_j^l \geq \cos\alpha_j^l$. Moreover, since there is only a finite number of different rhombi in R, $k = O(|b - w|)$. For the same reason, there is a finite number of angles β_j^l, and they are all in $[-\pi + \varepsilon, \pi + \varepsilon]$, for some small $\varepsilon > 0$ (in the general case where the angle of the vector $x_2 - x_1$ is θ_0, the angles β_j^l would be in the interval $[\theta_0 - \pi + \varepsilon, \theta_0 + \pi + \varepsilon]$). Thus by theorem 4.3 of [7], we have:

$$K^{-1}(b, w) = \frac{1}{2\pi}\left(\frac{1}{b - w} + \frac{\gamma}{\bar{b} - \bar{w}}\right) + \frac{1}{2\pi}\left(\frac{\xi_2}{(b - w)^3} + \frac{\gamma\bar{\xi_2}}{(\bar{b} - \bar{w})^3}\right) + O\left(\frac{1}{|b - w|^3}\right), \tag{14}$$

where $\gamma = e^{-i(\theta_1+\theta_2)} \prod_{j=1}^{k-1}\prod_{l=1}^{2} e^{i(-\beta_j^l+\alpha_j^l)}$, and $\xi_2 = e^{2i\theta_1} + e^{2i\theta_2} + \sum_{j=1}^{k-1}\sum_{l=1}^{2} e^{2i\beta_j^l} - e^{2i\alpha_j^l}$.

Note that for $j = 1, \ldots, k-1$, we have $\alpha_j^2 \equiv (\beta_j^1 + \pi) \bmod[2\pi]$, and $\beta_j^2 \equiv (\alpha_j^1 + \pi) \bmod[2\pi]$, thus:

$$\begin{aligned} \prod_{l=1}^{2} e^{i(-\beta_j^l+\alpha_j^l)} &= e^{i(-\beta_j^1+\alpha_j^1)} e^{i(-\alpha_j^1-\pi+\beta_j^1+\pi)} = 1, \\ \sum_{l=1}^{2} e^{2i\beta_j^l} - e^{2i\alpha_j^l} &= e^{2i\beta_j^1} - e^{2i\alpha_j^1} + e^{2i(\alpha_j^1+\pi)} - e^{2i(\beta_j^1+\pi)} = 0. \end{aligned}$$

Therefore $\gamma = e^{-i(\theta_1+\theta_2)}$, $\xi_2 = e^{2i\theta_1} + e^{2i\theta_2}$, which proves the theorem. $\square$

5.2 Asymptotics of the measures μ^R and μ

Let $R \in \mathcal{R}$ be a rhombus with diagonals tiling of the plane, and let R_d be the corresponding rhombus-with-diagonals tiling. Consider $\{w_1 b_1, \ldots, w_k b_k\}$ a subset of edges of ${R_d}^*$, and let E be the corresponding cylinder set of ${R_d}^*$.

Corollary 5.2 *When $|w_j - b_i| \to \infty$, $\forall\, j \neq i$, $\mu^R(E)$ only depends on the rhombi of R to which the vertices b_i and w_j belong, and else is independent of the structure of the graph R.*

Proof: This is a consequence of the explicit formula for $\mu^R(E)$ of theorem 3.2, and of the asymptotic formula for the inverse Dirac operator of theorem 5.1. $\square$

Recall $\Gamma^* = \cup_{L\in\mathcal{L}} {L_d}^*$. Let $\{e_1 = w_1 b_1, \ldots, e_k = w_k b_k\}$ be a subset of edges of Γ^*, and define $\mathcal{L}^E \subset \mathcal{L}$ to be the set of lozenge tilings of the plane that contain the lozenges associated to the edges $e_1, \ldots, e_k$.

Corollary 5.3 *When $|w_j - b_i| \to \infty$, $\forall\, j \neq i$, $\mu^L(E)$ is independent of $L \in \mathcal{L}^E$.*

Proof: As in section 4, we choose an embedding of Γ^* so that every edge of Γ^* uniquely determines the lozenge(s) it belongs to. Corollary 5.3 is then a restatement of corollary 5.2. $\square$

Let $\mathcal{E}$ be the cylinder set of Γ^* corresponding to the edges $e_1, \ldots, e_k$, and let F be the cylinder set of H (the honeycomb lattice) corresponding to the lozenges associated to the edges $e_1, \ldots, e_k$.

Corollary 5.4 *When $|w_j - b_i| \to \infty$, $\forall\, j \neq i$, and for every $L \in \mathcal{L}^E$, we have $\mu(\mathcal{E}) = \nu(F)\mu^L(E)$.*

Proof: This is a consequence of the explicit formula for $\mu(\mathcal{E})$ of theorem 4.1, and of corollary 5.3. $\square$

References

[1] P. Billingsley, Probability theory, *Wiley*, (1986).

[2] H. Cohn, R. Kenyon, J. Propp, A variational principle for domino tilings, *J. Amer. Math. Soc.*, **14** (2001), 297-346.

[3] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, Alternating sign matrices and domino tilings, *J. Algebraic Combin.* **1** (1992), 111-132.

[4] M. Fisher, H. Temperley, The dimer problem in statistical mechanics - an exact result, *Philos. Mag.* **6** (1961), 1061-1063.

[5] P.W. Kasteleyn, The statistics of dimers on a lattice, I. The number of dimer arrangements on a quadratic lattice, *Physica* **27** (1961), 1209-1225.

[6] P.W. Kasteleyn, Graph theory and crystal physics, *Graph Theory and Theoretical Physics, Academic Press, London* (1967), 43-110.

[7] R. Kenyon, The Laplacian and Dirac operators on critical planar graphs, *Invent. math.* **150** (2002), 409-439.

[8] R. Kenyon, Local statistics of lattice dimers, *Ann. Inst. H. Poincaré, Probabilités* **33** (1997), 591–618.

[9] R. Kenyon, A. Okounkov, S. Sheffield, Dimers and amoebas, (2003), math-ph/0311005.

[10] R. Kenyon, J-M. Schlenker, Rhombic embeddings of planar graphs, (2003), math-ph/0305057.

[11] G. Kuperberg, An exploration of the permanent-determinant method, *Electron. J. Combin.* **5** (1998), Research Paper 46, 34 pp. (electronic).

[12] C. Mercat, Exponentials form a basis of discrete holomorphic functions, (2002), math-ph/0210016.

[13] C. Mercat, Discrete period matrices and related topics, (2001), math-ph/0111043.

[14] J. Propp, Lattice structure for orientations of finite graphs, (1994), math.CO 0209005.

[15] W.P. Thurston, Conway's tiling groups, *Amer. Math. Monthly* **97** (1990), 757-773.